Life of Fred®

Abstract Algebra

Stanley F. Schmidt, Ph.D.

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A Note Before We Begin

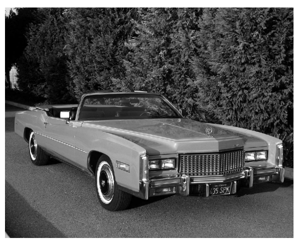
There are four algebra courses. Two of which, beginning algebra and advanced algebra, you took in high school. Linear algebra you will take sometime after this book.

This is abstract algebra. This book along with *Five Days of Upper Division Math* are usually the first two books taken as an upper division (junior/senior) student in college.

Be prepared for a very happy change.

Engineers filled many of the seats in Fred's lower-division calculus and statistics classes. He taught them how to find **answers** to stuff. 16 feet 7 pounds 3 hours 4 miles $88 6 cherry plums

The engineers need answers to build stuff.

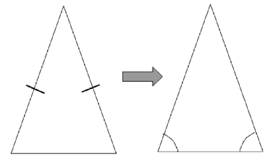


There were often 1700 students in Fred's calculus class. In his abstract algebra class there were 162 students. The engineers had left to go off and make things that make our lives nicer. Thank you engineers and goodbye.

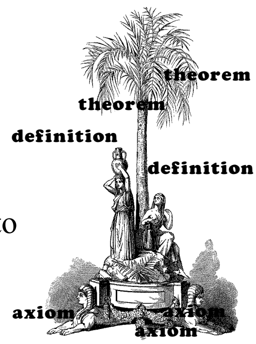
We mathematicians who remain have a dirty little secret. No more finding answers, no more producing stuff. We have only one goal. We are here to *play*. We invent fairy castles in the air that are stocked with pizza and gelato. We will play tag on the forest paths. If scientists find some of our creations useful to them, we are happy to share.

What mathematicians do is solve puzzles. Given a true statement, we find a proof. The high school course that was nearest to the spirit of mathematics was geometry. You were given an isosceles triangle and you proved that the base angles were equal.

(In geometry we found four different ways to prove that.)



In the Courtyard of Mathematics there are many statues, which celebrate the different fields of math: geometry, calculus, set theory, and so on. Each statue looks like a palm tree. At its base is the beginning assumptions (known as axioms or postulates). The tree grows as we prove statements (known as theorems).



a theory

Sometimes we add a definition. This is just to shorten things up. In geometry we defined a triangle ABC to be "the three segments $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$." It was much nicer to write ΔABC than to keep writing the three segments $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$.

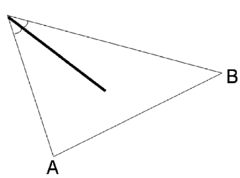
The whole palm tree—axioms, theorems, and definitions—is called a **theory**.

In high school we chose geometry as the first theory for students with since triangles and squares were things that they knew a little about already. They were real.

We had to introduce a zillion postulates to state the "facts" that we would start the growth of our geometry theory.

Postulate 1: One and only one line can be drawn through any two points.
Postulate 2: (The ruler postulate) Each point on a line can be uniquely matched up with a real number.
Postulate 3: (The angle measurement postulate) Every angle can be matched up with a number between 0 and 180.
Postulate 4: If two angles form a linear pair, then they are supplementary.
Postulate 5: (The angle addition postulate)
Postulate 6: (SSS) If three sides of one triangle are congruent to three sides of a second triangle, then the triangles are congruent.
Postulate 7: (SAS)
Postulate 8: (ASA)
Postulate 9: (The parallel postulate) There is at most one line through point P parallel to line ℓ.
Etc.

Even with that ton of axioms, there were lots of giant gaps that we never pointed out to the students. For example, given an angle bisector in a triangle, we could never prove that it hit the opposite side between A and B!



Worse yet, we could prove that every triangle is isosceles! (Done on pages 78–79 of *Life of Fred: Metamathematics*.)

In abstract algebra (the book you are holding in your hands right now) there is a big happy change. We won't calculate **answers** like we did in calculus. And no long list of postulates like high school geometry. A palm tree with just theorems and definitions. All *play*.

Contents

Chapter One
MESS Hall

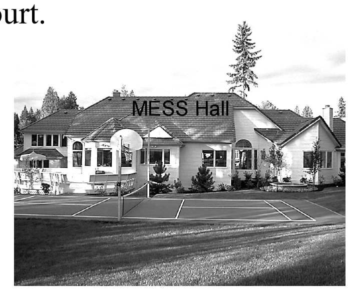
The freshmen students didn't know where to head. It was their first day on the KITTENS University campus. They had chosen KITTENS because of the wonderful math education that Fred offered. Word of this six-year-old's teaching ability had spread around the world.

One of the new students asked Joe where the new students for mathematics should meet. Joe was obviously one of the older students. His KITTENS T-shirt was faded.

Joe pointed toward the north end of the campus and said, "All the new students who like math meet in the MESS Hall at 9 a.m. today."

"Mess hall?" The new students were confused. A mess hall is where people go to eat. They didn't realize that Joe had said, "MESS Hall" not "mess hall."

They headed to the north end of campus. There it was—just beyond the tennis court.



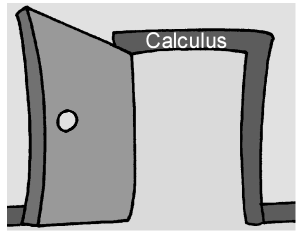
Fred greeted them as they came in. At 9 a.m. he began, "Welcome to the Mathematics Entrance Starting Station."

The students then realized what MESS stood for. They were in the right spot.

Inside the hall was one giant room. Each student was carrying a large backpack with a load of math from their pre-college days. They had two algebra courses (*Life of Fred: Beginning Algebra* and *Life of Fred: Advanced Algebra*), geometry (*Life of Fred: Geometry*), and trig (*Life of Fred: Trigonometry*). They already knew more than most people that you meet in everyday life.

On the walls of the giant room were doors, each leading to a different part of math—giant doors, big doors, and little doors.

Fred pointed toward one of the giant doors marked Calculus and said, "To enter calculus you're going to need almost everything in your math backpack.



For example, you'll need:

✓ the secant function from trig, since the derivative of the tan x is $\sec^2 x$.

✓ the binomial formula from algebra, since we'll use it to derive the derivative of $y = x^n$.

✓ factoring from algebra. We will want to stick x = 3 into $\frac{x^3 - 3x^2 + 6x - 18}{x - 3}$ (in Chapter 2) and the only way we can do it is to factor the numerator and cancel like factors.

✓ Σ notation.

✓ what a function is.

✓ and everybody's favorite collections of numbers:

$\mathbb{N} = \{1, 2, 3, \ldots\}$ (the natural numbers)

$\mathbb{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3 \ldots\}$ (the integers)

$\mathbb{Q}$ = all numbers that can be written as a fraction. Symbolically

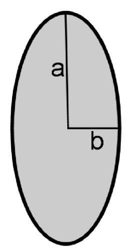
$\mathbb{Q} = \{p/q \mid p \in \mathbb{Z} \ \& \ q \in \mathbb{Z} \ \& \ q \neq 0\}$ (the rational numbers)

$\mathbb{R}$ = every number than can be written as an unending decimal. (the real numbers)

Those that repeat, such as 0.142857142857142857142857142857142857142857 14. . . , which equals $\frac{1}{7}$ are the rational numbers.

Those that don't repeat, such as 3908.89934256814154351778221039 69. . . , are the irrational numbers."

✓ lots of things from geometry. In calculus we'll find the area of an ellipse.* (*Life of Fred: Calculus, Chapter 11*)



Some of the students carried their heavy backpacks through the door marked Calculus.

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* It is A = πab. Many (most?) high school math teachers don't know this.

"On the other hand," Fred continued, "there are little doors. You won't have to carry your full backpack of two algebra courses, geometry, and trig.*

"One little door is Abstract Arithmetic. No need for any algebra. No need even for ℕ, ℚ, or ℝ. The only thing you need from set theory is some non-empty set X with lots of elements in it. Once you get past the front door, we're going to define a function f: X → X that has five properties. By the time we're done we will create ℕ, ℚ, and ℝ."

(*Life of Fred: Five Days of Upper Division Math*)

An Aside

Functions like f: X → X are called **unitary operations**. Unitary operations take a single element and give a single answer.

Squaring is a unitary operation. 7 → 49

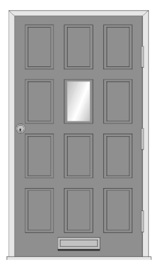
Factorial is a unitary operation. 5! → 120

"Your father is . . ." is a unitary operation. Name any person now living on earth and that person will have exactly one biological father.

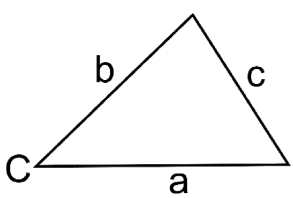
Some of the students loved that little door. They emptied out most of their backpacks and headed through that little door.

Fred pointed toward a medium-sized door and said, "Most of the doors leaving the MESS Hall are neither giant nor small. You don't have to remember everything like calculus requires—just some things. (*Life of Fred: Linear Algebra, LOF: Real Analysis, LOF: Complex Analysis, LOF: Metamathematics, LOF: Numerical Analysis, LOF: Logic*) In none of them, if I can remember right, will you never need to recall the generalization of the Pythagorean theorem: In any triangle

$$c^2 = a^2 + b^2 - 2ab\cos C.$$



regular-sized



* English lesson: When you have several paragraphs quoting the same person, only the last paragraph has an end quote. (")

"But in virtually all of the courses you will need to recall what you learned in high school about functions."

At this point Fred couldn't help himself. The idea of functions is simple but ethereal.*

You start with any two non-empty sets, say, A and B. A function from A to B is a rule which associates to each element of A exactly one element of B. f: A ➞ B

If A is the set of all dogs in San Francisco and B is ℝ (the real numbers), then one possible function might be the rule: *Assign to each dog the number of fleas it has on it right now.*

A second possible function might be the rule: *Assign to each dog its weight in pounds.*

If A = B, then f: A ➞ A is a unitary function (but you knew that already).

In high school algebra you graphed ordered pairs of numbers. You knew that (5, 8) is a different point than (8, 5). What you teacher may never have had the courage to mention is what ordered pairs are in relation to sets. You are going to need this to get through the door marked "Abstract Algebra."

Suppose that you have an ordered paid where the first coordinate is a student's first name and the second coordinate is a natural number ℕ. An order pair might look like (Chris, 83) or (Kim, 96969) or (Leslie, 2).

If S is the set all all student's first names, then these ordered pairs are members of S × ℕ. Seven facts . . .

① (Dale, 7) is a member of S × ℕ since Dale ∊ S and 7 ∊ ℕ.

② S × ℕ is called the **cross product** of S and ℕ.

③ A × B is defined as the set of all ordered pairs (a, b) where a ∊ A and b ∊ B. In symbols: A × B = { (a, b) | a ∊ A & b ∊ B }.

④ S × ℕ is also called the **cartesian product** of S and ℕ.

⑤ S × ℕ is not the same as ℕ × S. (Dale, 7) ∊ S × ℕ, but (Dale, 7) ∉ ℕ × S.

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* Pronounced i-THEER-e-el where the i is pronounced like the i in *if.* Ethereal objects are light and airy.

Another way of restating ⑤ is to say that × is not commutative.
⑥ The symbol × goes between two sets. Let's not confuse it with the multiplication of numbers.

But *the number of elements* of A × B is equal to *the number of elements* of B × A.

If A = {g, h, i} and B = {sofa, chair}, then A × B and B × A each contains six ordered pairs.
A × B = {(g, sofa), (h, sofa), (i, sofa), (g, chair), (h, chair), (i, chair)}
B × A = {(sofa, g), (sofa, h), (sofa, i), (chair, g), (chair, h), (chair, i)}

⑦ × is not associative. (A × B) × C ≠ A × (B × C).
For example if A = student's names and
B = ℕ and
C = {☎, ✂, ✐}
then ((Glen, 5), ✐) ∈ (A × B) × C, but ((Glen, 5), ✐) ∉ A × (B × C).*

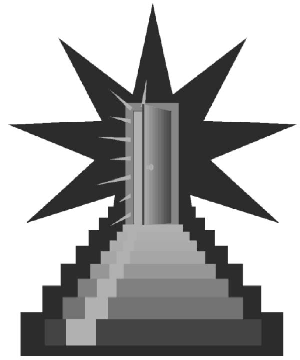
The reason you need to know about ordered pairs and cross products is to get through the Abstract Algebra door. You need to know about **binary operations**.

A unitary operation was f: A → A.

A binary operation is f: A × A → A. That's not really so mysterious. Addition is a binary operation. 44 + 32 = 76.
Multiplication is a binary operation. 4 times 7 equals 28.
Subtraction in the integers ℤ is a binary operation. 5 – 8 = –3.
With division we have to be a bit careful. Division in the positive rational numbers is a binary operation. Division in the natural numbers is *not* a binary operation since 5 ÷ 3 is not a natural number.

The students were getting a bit restless. They wanted to know where the Abstract Algebra door was. (If you bought this book, you probably also want to know!)

* But (Glen, (5, ✐)) ∈ A × (B × C).



Some of them were expecting a impressive doorway that required much of high school math.

(Not the Abstract Algebra door ☞)

Fred explained, "The truth is that there is no door to abstract algebra. No giant door, no big door, no little door. You don't even need to know everything that you learned about functions. Forget about one-to-one functions or onto functions. The only prerequisite is knowing what a binary operation is.

(And that you learned on the previous page of this book.)

"Forget about ℕ, ℚ, and ℝ.

"You don't need to remember how long it takes Sam and Sig to work together to dig a ditch if Sam can do the whole job in 4 hours and Sig can do the whole job in 6 hours.*

"This is *abstract* algebra—not the *applied* stuff of high school algebra."

One of the students became agitated and complained, "Wait a minute! What good is this abstract algebra stuff if you can't solve d = rt problems or how long it takes for my money to double in an investment that returns 8% per year?** It's gotta be useful."

Fred loved this question. It struck at the heart of what a real university education should be.

"Our world needs plumbers, and there are technical schools that can teach plumbing.

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* Let x = the number of hours it will take both of them to dig the ditch. Then in one hour they can do $\frac{1}{x}$ of the job.
Sam can do $\frac{1}{4}$ of the job in one hour. Sig can do $\frac{1}{6}$ of the job in one hour.

So in one hour $\frac{1}{x} = \frac{1}{4} + \frac{1}{6}$ etc.

** $(1.08)^x = 2$ Use logs to solve it.

"Our world needs secretaries, accountants, hair dressers, business owners, airplane mechanics, fashion designers, florists, truck drivers, insurance salesmen/women, ranchers, and postal workers, but . . .

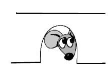
job training isn't what we do at KITTENS University.

KITTENS University is not spelled KITTEN$ University."

Fred opened the KITTENS catalog of courses. Each of these courses is designed to enlarge the student's view of the world. You study philosophy to understand the great questions of why. You study the great masterpieces of music, of art, of mathematics to gain an appreciation of the visions that the greatest minds produced.

When you pass the portal into abstract algebra (carrying only a knowledge of what a binary operation is), you will enter a playground where the only good is the pleasure of living in a phantasmical[*] world created by inspired algebraists.

With calculus you needed a giant door to carry in all the high school math. With abstract algebra the "door" is just a little mouse hole.



Students threw off their heavy backpacks and entered.

[*] I always wanted to use that word. A phantasm is a creation of the imagination.

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